Tiling deformations, cohomology, and orbit equivalence of tiling spaces

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Abstract

We study homeomorphisms of tiling spaces with finite local complexity (FLC), of which suspensions of d-dimensional subshifts are an example, and orbit equivalence of tiling spaces with (possibly) infinite local complexity (ILC). In the FLC case, we construct a cohomological invariant of homeomorphisms, and show that all homeomorphisms are a combination of tiling deformations, translations, and local equivalences (MLD). In the ILC case, we construct a cohomological invariant in the so-called weak cohomology, and show that all orbit equivalences are combinations of tiling deformations, translations, and topological conjugacies. These generalize results of Parry and Sullivan to higher dimensions. When the tiling spaces are uniquely ergodic, we show that homeomorphisms (FLC) or orbit equivalences (ILC) are completely parametrized by the appropriate cohomological invariants. We also show that, under suitable cohomological conditions, continuous maps between tiling spaces are homotopic to compositions of tiling deformations and either local derivations (FLC) or factor maps (ILC).

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1 Introduction

In this paper, we study orbit equivalences (or flow equivalences) between minimal, aperiodic tiling spaces with or without finite local complexity.

Tiling spaces in dimension d are compact spaces carrying an $\mathbb{R}^d$-action. A typical example is given by the suspension of a minimal $\mathbb{Z}^d$-subshift (for tilings with *finite local complexity*, *i.e.*, which have locally finitely many configurations), but examples without the FLC assumption can be much richer. The action of $\mathbb{R}^d$ gives the tiling space a foliated structure; an *orbit-equivalence* between two tiling spaces is a homeomorphism which preserves this structure, *i.e.*, which sends $\mathbb{R}^d$-orbits to $\mathbb{R}^d$-orbits. In the FLC case, the transversals of this foliated structure are totally disconnected, which guarantees that any homeomorphism is an orbit equivalence.

Previously, the second author [9] defined a special family of orbit equivalences between minimal tiling spaces of finite local complexity: tiling deformations (or shape-change deformations), and showed that they are related to the first cohomology group of the tiling space. The first author [15] showed that any orbit equivalence between two such tiling spaces defines an underlying deformation. As a consequence, it was proved that orbit equivalences can be classified by a cohomological invariant, up to topological conjugacy.

In the present paper, we strengthen these results along two lines. First, in the setting of FLC tiling spaces, we construct a sharper invariant than the one in [15]. We use it to classify homeomorphisms up to MLD (mutual local derivable) equivalence plus translation.[1] Second, we extend the formalism of tiling deformations to tilings without finite local complexity, and use another cohomological invariant to classify orbit equivalences up to topological conjugacy. In the same way that FLC tilings can model quasicrystals, non-FLC tilings (also called ILC for infinite local complexity) can be used to model less ordered materials, such as metallic glasses [5]. As mathematical objects, examples of such spaces are given by suspensions of homeomorphisms: if $f : X \to X$ is a minimal homeomorphism of a compact space, and $r : X \to \mathbb{R}_+$ is a continuous function, define

$$S^r X := \{(x,t) \in X \times \mathbb{R}^+ \ , \ 0 \leq t \leq r(x)\}/(x, r(x)) \sim (f(x), 0). \qquad (1)$$

If X is a minimal subshift over a finite alphabet, and r is locally constant, $S^r X$ can be seen as an FLC tiling space (the tiles have finitely many lengths). If X is any compact space and r is continuous, $S^r X$ can be seen as an ILC tiling space. The tilings spaces considered in this paper can have arbitrary dimension, and are d-dimensional analogues of the suspension described above.

The invariants used here are cohomology groups associated with the space, and have been studied previously. There are numerous descriptions of

[1] A local derivation is a continuous-time analogue of "sliding block codes" in the symbolic setting. A homeomorphism is an MLD map if it and its inverse are local derivations.

tiling cohomology in the literature, but it turns out that they all isomorphic to either the *strong* cohomology groups or the *weak* cohomology groups, depending on the regularity condition which is assumed of the cochains [7]. The groups that we use are the first cohomology groups with values in $\mathbb{R}^d$. It is convenient to describe these groups in terms of $\mathbb{R}^d$-cochains on Ω (inspired by groupoid cohomology). For the sake of completeness, Section 4 provides explicit isomorphisms between the different pictures of weak and strong cohomology.

To classify homeomorphisms in the FLC case, we use the strong cohomology group $H^1_s(\Omega;\mathbb{R}^d)$, which is known to be isomorphic to the Čech cohomology $\check{H}^1(\Omega;\mathbb{R}^d)$ [18, 9]. This group is an invariant of *tiling deformations*, also known as *shape changes*: consistent change in the shapes and sizes of the tiles that preserves the combinatorics of the tilings [9]. The upshot is that all homeomorphisms between FLC tiling spaces are a combination of translations, shape changes, and MLD equivalences. Since MLD equivalences and translations are cohomologically trivial, this invariant of deformations becomes an invariant of homeomorphisms.

When it comes to the description of flow equivalences in the setting of infinite local complexity (ILC), the relevant invariant is the first weak cohomology group with values in $\mathbb{R}^d$: to each orbit equivalence $\varphi : \Omega \to \Omega'$, we associate an element of $H^1_w(\Omega;\mathbb{R}^d)$. This invariant classifies orbit equivalences up to topological conjugacy and translation.

In dimension one, these results were essentially known: a theorem of Parry and Sullivan [21, 22] states that flow equivalence between two Markov shifts or minimal subshifts (*i.e.*, orbit-equivalence of their suspensions) is generated by a re-labeling of the tiles and a change in the length of the tiles (a mutual local derivation and a tiling deformation). More precisely, if X and Y are two minimal subshifts, and φ is a homeomorphism $SX \to SY$ (where $SX = S^1X$ is the suspension associated with the constant function $r = 1$), then there exists a locally constant function of constant sign r, defined on X, and a topological conjugacy $\Phi : S^rX \longrightarrow SY$. In particular, r defines an element in the first cohomology group

$$H^1(X,\mathbb{Z}) \otimes \mathbb{R} \simeq \Big(\{f : X \to \mathbb{Z}\}/\langle f - f\circ\sigma\rangle\Big) \otimes \mathbb{R},$$

where σ is the shift map. The class $[r]$ is then either a positive or negative element in the ordered cohomology group, depending whether r is positive or negative (and depending whether φ preserves or changes the orientation). Conversely, if $[f]$ is any element in $H^1(X,\mathbb{Z}) \otimes \mathbb{R}$ whose pairing with every invariant probability measure is (say) positive, then there exists a represen-

tative $f_0 > 0$ of this cohomology class [8, 12]. In this case, the suspension $S^{f_0}X$ is a well-defined object, and in our formalism, the invariant for the obvious map $SX \to S^{f_0}X$ is precisely $[f_0] = [f]$. Hence, non-infinitesimal elements of the first ordered cohomology group of X exactly parametrize flow equivalences from X to other subshifts, up to conjugacy.[2]

To define our invariant in higher dimensions, we define a distinguished class in the first cohomology of a tiling space with values in $\mathbb{R}^d$. We call this class the *fundamental shape class* and denote it $[\mathcal{F}]$ (or $[\mathcal{F}(\Omega)]$, when we want to make the space Ω explicit). In the cellular picture of cohomology, a tiling gives a CW-decomposition of $\mathbb{R}^d$, and this class is represented by a 1-cochain that assigns the vector $y - x$ to an edge between x and y. If $h : \Omega \to \Omega'$ is a homeomorphism, we define $[h] = h^*[\mathcal{F}(\Omega')]$.

For FLC tiling spaces, the difficulty is that there is a mismatch between our description of $[\mathcal{F}]$ and the settings where the pullback h^* is well-defined. The description involved pattern-equivariant cochains, but pullbacks of pattern-equivariant cochains only make sense when the map h sends transversals of Ω to transversals of Ω'. There are two ways around this difficulty, and we will explore both.

One approach is to invoke the isomorphism between the strong cohomology of an FLC tiling space and its Čech cohomology. The class $[\mathcal{F}(\Omega')] \in H^1_s(\Omega', \mathbb{R}^d)$ corresponds to a class in $\check{H}^1(\Omega', \mathbb{R}^d)$, which pulls back to a class in $\check{H}^1(\Omega; \mathbb{R}^d)$, which corresponds to a class in $H^1_s(\Omega; \mathbb{R}^d)$. This approach shows that $[h]$ is well-defined, but doesn't explain what information about h is carried in $[h]$.

A second approach is to find a map h_s, homotopic to h, such that h_s sends transversals of Ω to transversals of Ω'. We then define $[h]$, directly in strong pattern-equivariant cohomology, to be $[h_s^*\mathcal{F}(\Omega')]$, and prove that this does not depend on the choice of h_s.

In the ILC case, we use yet another description of tiling cohomology, involving dynamical cocycles. If h is an orbit equivalence, then for each tiling $T \in \Omega$ we associate the orbits of T and $h(T)$ with $\mathbb{R}^d$, and construct a homeomorphism $h_T : \mathbb{R}^d \to \mathbb{R}^d$ via the equation:

$$h(T - x) = h(T) - h_T(x).$$

This gives a dynamical cocycle, and we define $[h]$ to be the class of that cocycle in the *weak* cohomology of Ω.

[2] To be precise, the quotient of the cohomology group $H^1(X, \mathbb{Z}) \otimes \mathbb{R}$ by its infinitesimals parametrizes the locally constant suspensions of X up to conjugacy. The group itself parametrizes suspensions up to a finer equivalence (which we call MLD in the setting of tiling spaces).

In either setting, we prove that the class $[h]$ characterizes the homeomorphism h up to translation and an appropriate notion of equivalence:

Theorem 1.1 (Theorem 6.4)**.** *Let $h_i : \Omega \to \Omega_i$ be two orbit-equivalences ($i \in \{1, 2\}$).*

1. *If $[h_1] = [h_2]$ as elements of the weak cohomology, then there exists a continuous $s : \Omega \to \mathbb{R}^d$ such that $\tau_s : T \mapsto T - s(T)$ is a homeomorphism, and there exists a topological conjugacy $\varphi : \Omega_1 \to \Omega_2$ such that $h_2 \circ \tau_s = \varphi \circ h_1$.*

$$\begin{tikzcd} \Omega \arrow[r, "h_1"] \arrow[d, "\tau_s"'] & \Omega_1 \arrow[d, "\varphi"] \\ \Omega \arrow[r, "h_2"] & \Omega_2 \end{tikzcd}$$

2. *If the tiling spaces are FLC and $[h_1] = [h_2]$ in the strong cohomology, then the same statement as above holds, with φ an MLD map.*

In the special case that Ω and Ω' are in fact FLC tiling spaces, $[h]$ can be represented by the same cochain as in the FLC invariant described above, and is in fact the invariant developed in [15]. The upshot is that the ILC invariant carries less information than the FLC invariant, but is applicable in a wider setting.

Another question is what values the class $[h]$ may take. For uniquely ergodic tiling spaces we give a complete answer, both in the FLC and ILC settings, which extends the result known in dimension 1. Given an invariant measure μ, the Ruelle–Sullivan map C_μ sends elements of $H^1_s(\Omega; \mathbb{R}^d)$ or $H^1_w(\Omega; \mathbb{R}^d)$ to $d \times d$ square matrices. In one dimension, the image of a cocycle under the Ruelle–Sullivan map can be seen as an analogue of Poincaré's rotation number: a 1-cocycle valued in $\mathbb{R}$ can be restricted to an orbit and integrated to a 0-cochain[3], say F. Then by unique ergodicity,

$$\lim_{t \to +\infty} \frac{F(x+t) - F(x)}{t}$$

converges, and depends neither on x nor on the orbit. The matrix $C_\mu([h])$ describes the large-scale distortion associated with the orbit equivalence.

Theorem 1.2 (Theorems 7.1 and 8.1)**.** *Let $h : \Omega \to \Omega'$ be a homeomorphism between two uniquely ergodic tiling spaces (with or without finite local complexity). Then $C_\mu[h]$ is invertible. Furthermore, every cohomology class in*

[3] This is a 0-cochain of $\mathbb{R}$ and not of Ω, *a priori.* Cochains of Ω are assumed to be *pattern-equivariant*, see Section 4.

$H^1_w(\Omega;\mathbb{R}^d)$ (resp. $H^1_s(\Omega;\mathbb{R}^d)$, if Ω has FLC) whose image under the Ruelle–Sullivan map is invertible, is the class of some orbit equivalence of tiling spaces (resp. FLC tiling spaces).

In particular, for uniquely ergodic tiling spaces, the groups $H^1_{s/w}(\Omega;\mathbb{R}^d)$ not only provide an invariant for tiling spaces which are orbit-equivalent to Ω (up to topological conjugacy or MLD). They actually provide a *parametrization* of all such orbit equivalences.

As a remark, we mentioned that the Ruelle–Sullivan map captures the large scale behavior of a cocycle: given an orbit equivalence $[h]$ between two tiling spaces, then at large scale and on average, h maps orbit to orbit in a linear way. However, this does not mean that a cocycle α and its linear average $C_\mu[\alpha]$ stay at bounded distance from each other! The question of when this happens is left for another time, but we note that this problem was raised before, and even stated in terms of a cohomological equation in the framework of flows on the torus [11] and one-dimensional tiling spaces and laminations [1, 3, 4].

We also consider whether the assumption of unique ergodicity can be dropped. In the absence of unique ergodicity, we cannot construct nearly-constant cocycles to represent cohomology classes. Instead, we must make assumptions about the existence of representatives with properties (e.g., as in the definition of "small, positive cocycles" in [13]) that allow them to define tiling deformations. It is unclear to us how this would work: consider the one-dimensional case of the suspension of a subshift (Equation (1)). A function $r : X \to \mathbb{R}$ (assume it is locally constant) defines an element in $H^1(X,\mathbb{Z})\otimes\mathbb{R}$, and its image under C_μ is non-zero for every invariant measure if and only if $[r]$ is not an infinitesimal, that is there exists r_0 of constant sign representing $[r]$. In this case, $S^r X$ makes sense, and the homeomorphism between S^1X and S^rX has class $[r]$. However, it is unclear how such a description would generalize to higher dimensions: assume $C_\mu[\alpha]$ is a non-singular matrix for every invariant ergodic measure μ (assume, for example, $\det C_\mu[\alpha] > 0$); it is not even guaranteed that $C_\nu[\alpha]$ would be non-singular for an invariant measure $\nu = \mu_1 + \mu_2$ where μ_i are ergodic.

Finally, we consider cohomological invariants for arbitrary maps between tiling spaces that preserve translational orbits. We define a class $[h] \in H^1_w(\Omega;\mathbb{R}^d)$ for an arbitrary orbit-preserving map $h : \Omega \to \Omega'$ of tiling spaces, and a class $[h] \in H^1_s(\Omega;\mathbb{R}^d)$ when the spaces have FLC. We then prove the following analogue of Theorem 6.4:

Theorem 1.3. *Let $h_1 : \Omega \to \Omega_1$ be an orbit-equivalence, and let $h_2 : \Omega \to \Omega_2$ be an orbit-preserving continuous map.*

1. *If $[h_1] = [h_2]$ as elements of the weak cohomology, then h_2 is homotopic to a composition $\varphi \circ h_1$, where $\varphi : \Omega_1 \to \Omega_2$ is a factor map.*

2. *If the tiling spaces are FLC and $[h_1] = [h_2]$ in the strong cohomology, then the same statement as above holds, with φ a local derivation.*

If Ω is uniquely ergodic, and if $C_\mu[h_2]$ is invertible, then Theorem 1.2 guarantees the existence of a tiling deformation h_1 such that $[h_2] = [h_1]$. Under those conditions, h_2 is homotopic to the composition of a tiling deformation and a factor map (ILC) or local derivation (FLC).

The structure of the paper is as follows. In Sections 2–3 we review the formalism of tiling spaces and spaces of Delone sets, and in Section 4 we review the different notions of tiling space cohomology and the relations between them. In Section 5 we define the invariant $[h]$ for homeomorphisms of FLC tiling spaces and compute it in a number of examples. In Section 6 we define the invariant in $H^1_w(\Omega; \mathbb{R}^d)$ for orbit equivalences of general tiling spaces, and show how it relates to the previously defined invariant when the tiling spaces have FLC. We also show how the invariant classifies homeomorphisms up to translation and topological conjugacy (or MLD equivalence, depending on the setting). In Section 7 we define the Ruelle–Sullivan map and show that $C_\mu([h])$ is always invertible. In Section 8 we show that all homeomorphisms between tiling spaces are a combination of topological conjugacies (or, for FLC tiling spaces, MLD maps), translations, and “shape deformations”, the last being maps in which the shapes and sizes of the tiles are deformed while the combinatorics of the tiling are preserved. In Section 9 we demonstrate, by example, how H^1_w is typically infinite-dimensional, and how an ILC tiling space can be homeomorphic to an FLC tiling space without being topologically conjugate to any FLC tiling space. Finally, in Section 10 we prove Theorem 1.3.

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2 Tilings and Delone sets

In this section, we define the main objects which will be studied in the next parts of the paper, namely tilings and (labeled) Delone sets. For some applications, such as constructing transversals, it is more convenient to work with Delone sets. For others, such as pattern-equivariant cohomology, it is

more convenient to work with tilings. However, the two constructions are essentially equivalent.

Definition 2.1. A Delone set of $\mathbb{R}^d$ is a subset $\Lambda \subset \mathbb{R}^d$ which is *uniformly discrete* and *relatively dense*, in the sense that there exist respectively $r > 0$ such that any two points of Λ are at least at distance r apart; and $R > 0$ such that any ball of radius R in $\mathbb{R}^d$ intersects Λ.

A *labeled Delone set* is (the graph of) a map $\ell : \Lambda \to X$ from a Delone set to a compact metric space X, called the space of labels. Its elements are pairs $(\lambda, \ell(\lambda))$ where λ is a point of $\mathbb{R}^d$ and $\ell(\lambda)$ is its label. By abuse of terminology, we still call a labeled Delone set simply Λ.

Definition 2.2. A tile is a compact subset of $\mathbb{R}^d$ that is homeomorphic to a closed ball of positive radius. For simplicity, we will assume that a tile is a convex polytope. Tiles that are translates of one another are said to be equivalent. We pick a distinguished representative from each equivalence class, called a *prototile*. The set of prototiles can be given a topology induced by the Hausdorff distance on tiles. Given a compact set of prototiles $\mathcal{A}$, a tiling with tiles in $\mathcal{A}$ is a set $T = \{t_i\}_{i \in I}$ where the t_i are tiles whose class is in $\mathcal{A}$, the union of the t_i is all of $\mathbb{R}^d$, and any two distinct tiles only intersect on their boundary.

Similarly, we can define a labeled tiling as the graph of a map from T to a compact metric space of labels X. The set of labeled prototles is then a compact subset of $\mathcal{A} \times X$. If X is infinite, we also assume that the shapes and positions of the prototiles are such that convergence in X implies convergence in $\mathcal{A} \times X$.

Given a labeled Delone set, one can construct a tiling by looking at the Voronoi cells of the points of the Delone set, *i.e.*, at the regions closer to a particular point than to all others. Conveniently, Voronoi cells are always convex polytopes. One can also take the dual tiling to the Voronoi tiling, thereby obtaining a tiling whose vertices are the points of the original Delone set.

Conversely, given a tiling, we can construct a Delone set as follows. To each labeled tile (t, l) we associate a distinguished point x_t, called a puncture (*e.g.*, at the center of mass), and consider the labeled point $(x_t, ([t], l))$ where $([t], l) \in \mathcal{A} \times X$. Therefore, problems worded in terms of tilings can be reworded in terms of Delone sets, and vice versa. We use the generic term “tiling space” to refer either to a space of tilings or a space of Delone sets.

A *patch* or *local configuration* is the restriction of a tiling or Delone set to a bounded region. Formally, if Λ is a (labeled) Delone set, $\Lambda \cap B(x, R)$

is the restriction of Λ to the ball of radius R and center x. For a tiling, $T \cap B(x, R)$ is the set of all (decorated) tiles which intersect $B(x, R)$.

The set of tilings (or Delone sets) can be given a metric topology. For Delone sets, it coincides with vague convergence of measures, when Λ is identified with the sum of Dirac measures with support Λ. Note that the Delone set property implies that whenever Λ_n converges to Λ vaguely, any point x in Λ corresponds to a point $x_n \in \Lambda_n$ for n large enough, and $x_n \to x$. This justifies the following definition for convergence of labeled Delone sets.

Definition 2.3. Let Λ, Λ' be two labeled Delone sets, with the same space of labels X. We say that the distance between Λ and Λ' is less than ε if there exist a one-to-one map $\varphi : \Lambda \cap B(0, \varepsilon^{-1} - \varepsilon) \to \Lambda' \cap B(0, \varepsilon^{-1})$ and a one-to-one map $\varphi' : \Lambda' \cap B(0, \varepsilon^{-1} - \varepsilon) \to \Lambda \cap B(0, \varepsilon^{-1})$ which are inverse of each other (where defined), and such that

$$\forall x, x', \ \|\varphi(x) - x\| + \|\varphi'(x') - x'\| + D(\ell(x), \ell(\varphi(x)) + D(\ell(x'), \ell(\varphi'(x')) < 2\varepsilon,$$

where D is the distance on the space of labels.

That's a very complicated definition! However, the idea behind it is simple. Two Delone sets are close if on a large ball (of size $1/\varepsilon$) around the origin, the point-sets are very close; there is a one-to-one correspondence between points, corresponding points are ε-close in $\mathbb{R}^d$ and their labels are ε-close in X. Similarly, two tilings T and T' are close if on a large ball around the origin, they are nearly the same, in that there is a one-to-one correspondence between tiles, and corresponding tiles are close both in shape for the Hausdorff distance and in label.

Definition 2.4. A tiling T or Delone set Λ has *finite local complexity* (or FLC) if it has are only finitely many local configurations of a given size, up to translation. Otherwise, it has infinite local complexity (or ILC).

The metric for FLC tilings has a much simpler description: Λ and Λ' are within distance ε if there exists x, x' of norm less than ε such that $(\Lambda - x) \cap B(0, \varepsilon^{-1}) = (\Lambda' - x') \cap B(0, \varepsilon^{-1})$.

The group $\mathbb{R}^d$ acts on tilings and Delone sets by translation. Given a Delone set Λ (or a tiling), one can define its *hull* as the closure

$$\Omega_\Lambda := \overline{\{\Lambda - x \ ; \ x \in \mathbb{R}^d\}}.$$

Elements in the closure can still be interpreted as Delone sets (resp. tilings). This space supports an action of $\mathbb{R}^d$ by translations, and is a compact dynamical system. Our focus will be on *minimal* spaces, meaning that every $\mathbb{R}^d$-orbit is dense.

3 Tiling equivalences and transversals

We defined so far two families of tiling spaces: FLC tiling spaces and ILC tiling spaces. We now define a family of maps between tilings spaces, which can be seen as equivalence between the dynamical systems. FLC tiling spaces have additional structure, so it makes sense to define a family of more rigid maps between FLC tiling spaces.

Definition 3.1. A *factor map* $\varphi : \Omega \to \Omega'$ is a continuous map that satisfies

$$\varphi(T - x) = \varphi(T) - x.$$

In particular, it sends orbits to orbits. A *topological conjugacy* is a factor map that is a homeomorphism.

Definition 3.2. Let Ω and Ω' be FLC tiling spaces. A *local map* from Ω to Ω' a continuous map which satisfies $\forall r > 0,\ \exists R > 0,\ \forall T, T' \in \Omega$,

$$\big(T \cap B(0, R) = T' \cap B(0, R)\big) \Rightarrow \big(\varphi(T) \cap B(0, r) = \varphi(T') \cap B(0, r)\big).$$

A *local derivation* is a local map that is also a factor map. A *mutual local derivation* (or MLD map) is a local derivation that is a homeomorphism, and whose inverse is a local derivation. In particular, it is a conjugacy.

Transversals can be defined for all tiling spaces, but we will require more structure from transversals of FLC tiling spaces.

Definition 3.3. Let Ω be a space of Delone sets. The *canonical transversal* of Ω is the set

$$\Xi = \{\Lambda - \lambda \ ;\ \lambda \in \Lambda\}.$$

That is, it is the set of all Delone sets in Ω which have a point at the origin.

Definition 3.4. Let Ω be a tiling space (or a space of Delone set). A transversal for Ω is given by a factor map $\mathcal{D} : \Omega \to \Omega'$ to a space of Delone sets, and is defined by

$$\Xi_{\mathcal{D}} = \mathcal{D}^{-1}(\Xi'),$$

where Ξ' is the *canonical* transversal of Ω'.

When Ω has FLC, we define an *FLC-transversal* (or simply transversal when it is clear from the context) identically, except that the factor map $\mathcal{D}$ is assumed to be a local derivation.

We call the factor map $\mathcal{D}$ a "pointing rule" for the tilings of Ω.

Example 3.5. As a typical example, if Ω is a tiling space, let $\mathcal{D}(T)$ be defined as the Delone set of all barycenters of tiles in T. Then the associated transversal $\Xi_{\mathcal{D}}$ is often called in the literature "the canonical transversal of Ω". When Ω has FLC, this transversal is an FLC-transversal.

Tiling spaces have a local product structure in which one of the factors is given by $\mathbb{R}^d$, the group acting by translations. The following result states a known fact: tiling spaces (with or without FLC) are examples of foliated spaces (also called laminations).

Proposition 3.6. *Let Ω be a minimal aperiodic tiling space, with or without FLC. Then for all T, there exists a transversal Ξ containing T and $C > 0$ such that the map*

$$\Xi \times B(0, C) \to \Omega \ ; \ (T', x) \mapsto T' - x$$

is a homeomorphism onto its image (in particular its image is open).

Proof. Assume Ω is a space of Delone sets, and $0 \in T$. Let r, R be the Delone constants. Then, if Ξ is the canonical transversal and $C = r/2$, it is easy to check the conclusions of the theorem. If $0 \notin T$, let Ξ be the transversal given by the pointing rule $\mathcal{D}(T') = T' - x$, where x is any given point of T. □

In the FLC case, we have more rigidity.

Proposition 3.7 ([6])**.** *Let Ω be an aperiodic, minimal tiling space with finite local complexity. Then for all $T_0 \in \Omega$, there exist $\varepsilon > 0$ and an FLC transversal Ξ containing T such that the map*

$$B(0, \varepsilon) \times \Xi \longrightarrow \Omega \ ; \ (x, T) \longmapsto T - x$$

is a homeomorphism onto its image. In addition, Ξ is a Cantor set (compact, totally disconnected and with no isolated points).

It implies in particular that for FLC, aperiodic and minimal tiling spaces, the path-connected components are exactly the $\mathbb{R}^d$-orbits.

Finally, local maps can be characterized in terms of FLC transversals.

Proposition 3.8. *Let $f : \Omega \to \Omega'$ be a continuous map between FLC tiling spaces. It is a local map if and only if the image of any FLC transversal is included in an FLC transversal*

Proof. Let f be a local map between FLC tiling spaces. Let Ξ be an FLC transversal. It means that there is a local pointing rule $\mathcal{D}$ such that $\mathcal{D}(T)$ is a Delone set locally derived from T, for all T. We assume that there is a local configuration P (of the form $T_1 \cap B(0, R_1)$ for some tiling T_0), such that $0 \in \mathcal{D}(T) \Leftrightarrow T \cap B(0, R_1) = P$. (Here, R_1 is a constant of locality for the pointing rule.) There is no loss of generality: by finite local complexity, any FLC transversal is a finite union of such transversals. Let R_2 be the constant of locality for the map f such that for all tilings T_1, T_2,

$$T_1 \cap B(0, R_2) = T_2 \cap B(0, R_2) \Rightarrow f(T_1) \cap B(0, 1) = f(T_2) \cap B(0, 1).$$

By FLC, there are finitely many patches $Q_1, \ldots, Q_j$ of radius R_2 which extend P, that is $Q_i \cap B(0, R_1) = P$. Therefore, by locality of f, there are finitely patches of the form $Q'_i = f(T) \cap B(0, 1)$, for $T \in \Xi$. Define the following pointing rule on Ω':

$$\begin{aligned} 0 \in T' &\Longleftrightarrow \exists T \in \Xi,\ T' \cap B(0,1) = f(T) \cap B(0,1) \\ &\Longleftrightarrow \exists i,\ T' \cap B(0,1) = Q'_i. \end{aligned}$$

This pointing rule defines an FLC transversal Ξ' which contains $f(\Xi)$.

Conversely, assume f is continuous, and the image of any FLC transversal is included in an FLC transversal. Let $T_0 \in \Omega$. Define a pointing rule on Ω by $0 \in \mathcal{D}(T')$ if and only if $T' \cap B(0,1) = T_0 \cap B(0,1)$. Then, the image of Ξ is included in an FLC transversal Ξ', associated with a pointing rule $\mathcal{D}'$. The map f induces a continuous map $\Xi \to \Xi'$. It is uniformly continuous, and given how the topology restricts on FLC transversals, uniform continuity reads:

$$\begin{aligned} &\forall r > 0,\ \exists R > 0,\ \forall T_1, T_2 \in \Xi, \\ &\big(T_1 \cap B(0,R) = T_2 \cap B(0,R)\big) \Longrightarrow \big(f(T_1) \cap B(0,r) = f(T_2) \cap B(0,r)\big), \end{aligned}$$

which is exactly the locality condition for f. $\square$

4 Tiling cohomologies

There are several cohomology theories for a tiling space Ω, which mostly yield isomorphic cohomology groups. To begin with, we may consider the Čech cohomology $\check{H}^*(\Omega; A)$ with values in an arbitrary Abelian group A. This is computed from the combinatorics of open covers of Ω, and is manifestly a homeomorphism invariant. The trouble with Čech cohomology is

that it is difficult to relate the cohomology of the tiling *space* to properties of individual *tilings*.

Next there is the (strong) *pattern-equivariant* cohomology of Ω with values in A, denoted $H^*_{PE,s}(\Omega; A)$. Let T be an FLC tiling of $\mathbb{R}^d$ whose orbit is dense in Ω. The tiling T gives a decomposition of $\mathbb{R}^d$ as a CW complex. The 0-cells are the vertices of the tiling, the 1-cells are the edges, and so on.

A k-cochain α_k on T is an assignment of an element of A to each k-cell of T. α is called (strongly) *pattern equivariant* (or PE) if there exists a distance R such that the value of α on each cell depends only on the patch of radius R around the cell. That is, if there are cells c_1 and c_2 centered at points x_1 and x_2, and if the patch of radius R centered around x_1 matches the patch of radius R centered at x_2, then $\alpha(c_1) = \alpha(c_2)$. It is not hard to see that the coboundary of a PE cochain is PE (with perhaps a slightly different radius), so the PE cochains form a differential complex. $H^*_{PE,s}(T; A)$ is the cohomology of this complex. It is known [16, 9] that $H^*_{PE,s}(T; A)$ is canonically isomorphic to $\check{H}^*(\Omega; A)$. Thus, if Ω is a minimal tiling dynamical system, then its Čech cohomology may be computed from the pattern equivariant cohomology of *any* tiling in Ω. Moreover, PE cochains on any one tiling T extend by continuity to PE cochains on all other tilings in Ω. We can thus speak of the complex of PE cochains on Ω, rather than on a specific tiling, and denote the cohomology $H^*_{PE,s}(\Omega; A)$.

When A is a vector space, PE cohomology can also be defined using differential forms. (Indeed, this is how PE cohomology was first defined [16, 18].) One considers PE functions and PE differential forms with values in A, and constructs the de Rham complex. This yields a cohomology $H^*_{dR,s}(\Omega; A)$ isomorphic to $H^*_{PE,s}(\Omega; A)$, the version of PE cohomology based on cochains. Indeed, given a PE k-form, we can construct a PE cochain by integrating the form over k-cells. Conversely, given a PE cochain we can obtain a PE k-form whose integral is that cochain, *e.g.*, by linearly interpolating across tiles. These operations give explicit isomorphisms between the form-based and cochain-based PE cohomologies.

When A is a vector space, we can also consider *weakly* PE cochains or forms, which are defined to be uniform limits of strongly PE cochains or forms. (For forms we must also assume that the derivatives of the forms of all orders are uniform limits of PE functions.) This gives a cohomology theory that we call *weak* PE cohomology, denoted $H^*_{PE,w}$ or $H^*_{dR,w}$. This usually is *not* isomorphic to the Čech cohomology of Ω. Rather, it corresponds to the "dynamical" Lie-algebra cohomology of $\mathbb{R}^d$ with values in the space of continuous functions on Ω [18]. $H^*_{PE,w}$ is usually infinitely generated, even

for very simple tiling spaces, as will be illustrated in Section 9.

When the tilings have infinite local complexity, then it is unreasonable to expect patches to be identical, and strong pattern equivariance is not usually well-defined. However, a version of weak pattern equivariance always makes sense. We present this using an adaptation of groupoid cohomology, as follows. For definiteness we will take $A = \mathbb{R}^d$, which is the most important example for this paper. For the purpose of this article and to avoid lengthly discussion, we focus on the group H^1. Let $T \in \Omega$ be a given tiling in a tiling space Ω (which may or may not have finite local complexity).

Definition 4.1. If T has finite local complexity, a continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ is called *strongly T-equivariant* (or simply *pattern equivariant* or PE) if there exists a radius R such that $f(x) = f(y)$ whenever $T \cap B(x, R) = T \cap B(y, R)$ (up to translation).

Given a tiling T with or without finite local complexity, a continuous function f is called *weakly PE* if $f(x_n) \to f(x)$ whenever x_n is a sequence satisfying $T - x_n \to T - x$ in the tiling space of T. When T has FLC, weakly PE functions are a uniform limits of strongly PE functions.

Definition 4.2. A T-equivariant weak (resp. strong) dynamical 1-cocycle with values in $\mathbb{R}^d$ is a function $\beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfies

- *Cocycle condition:* $\beta(x, v) + \beta(x + v, w) = \beta(x, v + w)$;
- *Pattern equivariance:* for all v, the function $x \mapsto \beta(x, v)$ is weakly (resp. strongly) pattern equivariant.

This cocycle is a *weak (resp. strong) 1-coboundary* if there exists a continuous, weakly (resp. strongly) pattern-equivariant function $s : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, $\beta(x, v) = s(x + v) - s(x)$.

Note that strong 1-cocycles can be weak coboundaries, even when they are not strong coboundaries.

Remark 4.3. The weak pattern equivariance condition is a continuity condition for an appropriate topology: it is possible to embed $\mathbb{R}^d \times \mathbb{R}^d$ in $\Omega \times \mathbb{R}^d$ *via* the map $(x, v) \mapsto (T - x, v)$. By minimality, this embedding has dense image. A function defined on $\mathbb{R}^d \times \mathbb{R}^d$ will then extend continuously to a function on $\Omega \times \mathbb{R}^d$ if and only if it is weakly PE. Similarly, a PE 0-cochain can be seen as a continuous function on Ω.

As a consequence of this remark, it is possible to describe these PE cochains as follows:

- a weak PE 1-cocycle $\alpha : \mathbb{R}^d \times \mathbb{R}^d$ corresponds to a continuous function (still called α by abuse of notation) $\alpha : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\alpha(T, x) + \alpha(T - x, y) = \alpha(T, x + y)$;

- a weak 0-cochain corresponds to a continuous function $s : \Omega \to \mathbb{R}^d$;

- the co-boundary of a 0-cochain in this picture is defined by $\delta s(T, v) = s(T - v) - s(T)$.

This shows that the groups of cocycles and coboundary do not depend on the particular T used to define pattern-equivariance.

The notion of strong pattern equivariance is a bit more delicate. Strongly pattern equivariant functions on $\mathbb{R}^d$ correspond to continuous functions on Ω that are *transversally locally constant*, in the sense that their restriction to any FLC transversal is locally constant. (This definition requires in particular that T has FLC.)

Definition 4.4. Given a tiling space Ω, its dynamical 1-cohomology group $H^1_{d,w}(\Omega; \mathbb{R}^d)$ (resp. $H^1_{d,s}(\Omega; \mathbb{R}^d)$) is the quotient of the space of weak (resp. strong) dynamical 1-cochains by the space of equally muscular dynamical 1-coboundaries.

It will be convenient to define cohomology by using slightly smaller groups. Let Ω be a tiling space and Ξ a transversal. Let $\mathcal{D}$ be the associated pointing rule. Let T be a fixed tiling, so that $\mathcal{D}(T)$ is a Delone set. Let $(\mathbb{R}^d \times \mathbb{R}^d)_{|\Xi}$ be the set of all $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $x \in \mathcal{D}(T)$ and $x + v \in \mathcal{D}(T)$. We call *restriction to the transversal* Ξ the restriction of 1-cochains to this subset. Similarly, PE 0-cochains can be restricted to $\mathcal{D}(T)$. When dealing with strongly PE cochains, we require that Ξ is an FLC transversal, so that $\mathcal{D}(T)$ has FLC.

The following result shows that we can use either cochains on $\mathbb{R}^d \times \mathbb{R}^d$ or restricted cochains to represent an element in $H^1_{d,s/w}(\Omega; \mathbb{R}^d)$

Proposition 4.5. *The restriction of* 1*-cochains on a transversal induces an isomorphism between the dynamical cohomology group* $H^1_{d,s/w}(\Omega; \mathbb{R}^d)$ *of cochains on* $\mathbb{R}^d \times \mathbb{R}^d$*, and the group defined similarly with restricted cochains.*

Proof. We show the isomorphism for weak cochains. The strongly PE case is similar. Assume T is fixed, and cochains are T-equivariant. Without loss of generality, assume $0 \in \mathcal{D}(T)$. Clearly a 1-cochain (resp. coboundary) defined on $\mathbb{R}^d \times \mathbb{R}^d$ restricts to a cochain (resp. coboundary). So there is a well-defined map from the associated cohomology groups.

This map is onto: let β be a restricted 1-coboundary. Then by [17, Lemma 2.6, (3)], there exists $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\varphi(\lambda) = \beta(0, \lambda)$ for all $\lambda \in \mathcal{D}(T)$. Furthermore, it is quickly checked from Kellendonk's result that $\alpha(x, v) := \varphi(x + v) - \varphi(x)$ satisfies the condition for being a pattern-equivariant 1-cochain. So α is a 1-cochain defined on $\mathbb{R}^d \times \mathbb{R}^d$ which restricts to β.

The map is one-to-one: this is a similar argument on coboundaries using [17, Lemma 2.6, (1)]. □

Theorem 4.6. *The weak cohomology groups $H^1_{PE,w}(\Omega; \mathbb{R}^d)$, $H^1_{dR,w}(\Omega; \mathbb{R}^d)$ and $H^1_{d,w}(\Omega; \mathbb{R}^d)$ are isomorphic. Similarly, the strong cohomology groups $H^1_{PE,s}(\Omega; \mathbb{R}^d)$, $H^1_{dR,s}(\Omega; \mathbb{R}^d)$ and $H^1_{d,s}(\Omega; \mathbb{R}^d)$ are isomorphic.*

Proof. Let T be fixed, and be the reference for which pattern-equivariance is defined. T gives a CW decomposition of $\mathbb{R}^d$ in which an oriented edge e can be described by an origin x and a vector v, and we write $e = (x, v)$ (so that $-e = (x + v, -v)$). Given a dynamical 1-cocycle $\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, define a cellular 1-cochain by $\beta(e) = \alpha(x, v)$. It is an exercise to show that β is indeed a (cellular) cocycle, and is PE.

Conversely, if β is a cellular 1-cocycle, let us define a dynamical 1-cocycle restricted to the Delone set $\mathcal{D}(T)$ of all 0-cells of T. If x and $x+v$ are in $\mathcal{D}(T)$, there is a finite path of edges with appropriate orientations $\{e_i\}_i = \{(x_i, v_i)\}_i$ such that $x_1 = x$, $x_i + v_i = x_{i+1}$ and $\sum_i v_i = v$. Then define $\alpha(x, v) = \sum_i \beta(e_i)$. By the cellular cocycle condition, this sum is independent of path and only depends on (x, v). The resulting map α is a dynamical (restricted) 1-cocycle which is PE with respect to the first variable. It is weakly or strongly PE depending on β.

An identification of the 0-cochains and of the boundary map is checked similarly. □

Remark 4.7. In the discussion above, we described the groupoid cohomology of $\Omega \rtimes \mathbb{R}^d$ (or its reduction on a transversal Ξ) using two resolutions: one involving continuous functions and one involving transversally locally constant functions. It is an element of folklore that these cohomologies are isomorphic to the already investigated weak and strong cohomologies—not just in dimension 1. For example, one can see Ω as a model for the classifying space of the reduction of $\Omega \rtimes \mathbb{R}^d$ to Ξ [14]. However, caution should be exercised as there is more than one groupoid cohomology theory and not all theories give isomorphic cohomology groups for a groupoid and its reduction. This is why we gave the explicit isomorphisms at least for H^1.

In light of Theorem 4.6, we will henceforth usually write $H^1_s(\Omega;\mathbb{R}^d)$ and $H^1_w(\Omega;\mathbb{R}^d)$ for the strong and weak first cohomology groups, keeping in mind that we have many pictures for describing these groups.

The following result will be used a few times. In one-dimensional tiling spaces, it is essentially the classical Gottchalk-Hedlund Theorem. In higher dimensions, it can be deduced from generalizations of the Gottschalk-Hedlund Theorem. A direct proof in a related setting (with FLC tilings and cochains that are assumed to be closed and strongly PE) may be found in [19].

Proposition 4.8. *A* 1*-cocycle* α *is a* 1*-coboundary if and only if it is bounded. In the terminology of dynamical cohomology, a* 1*-cocycle is a* 1*-coboundary if and only if for some* x *(equivalently, for all* x*), the map* $v \mapsto \alpha(x,v)$ *is bounded.*

We end this discussion on cohomology by providing an explicit isomorphism between the dynamical H^1 groups and the de Rham H^1 groups which Kellendonk defined using pattern-equivariant forms.

Proposition 4.9. *Let* ω *be a closed, pattern-equivariant* 1*-form on* $\mathbb{R}^d$*. Let* f *be a (*a priori *not PE) function on* $\mathbb{R}^d$ *such that* $df = \omega$*. Then, the function* $\alpha(x,v) = f(x+v) - f(x)$ *is a* 1*-cocycle in the groupoid picture. It is weakly/strongly PE if* ω *is, and the map* $\omega \mapsto \alpha$ *induces an isomorphism in cohomology.*

Proof. We only remark that for any dynamical 1-cochain α, one can build

$$\beta(x,v) := \int_{\mathbb{R}^d} \alpha(x, v-s)\rho(s)ds,$$

where ρ is a compactly supported, positive function of class C^∞ and of integral 1. It is a 1-cochain, which is differentiable with respect to its second variable. It is easily checked that $v \mapsto \alpha(x,v) - \beta(x,v)$ is bounded, and therefore α and β lie in the same cohomology class by the result above. Now, let $\varphi(v) = \beta(0,v)$. Then $d\varphi$ is a PE 1-form, and $\beta \mapsto d\varphi$ is an inverse in cohomology of the map $\omega \mapsto \alpha$. $\square$

5 An invariant for homeomorphisms of FLC tiling spaces

We now define the *fundamental shape class* of a tiling space. If T is an FLC tiling, then there is a strongly PE 1-cochain Δx that assigns to each

oriented edge the displacement along that edge. This is manifestly closed, and represents a class in $H^1_s(\Omega, \mathbb{R}^d)$. In terms of dynamical cocycles, Δx corresponds to the function $\mathcal{F}(x,v) = v$. The cocycle $\mathcal{F}(x,v) = v$ is well-defined for arbitrary tiling spaces, not just for FLC tilings.

Definition 5.1. If Ω is an FLC tiling space, the fundamental shape class in $H^1_s(\Omega; \mathbb{R}^d)$ is the class represented by the 1-cochain Δx. If Ω is any tiling space, the fundamental shape class in $H^1_w(\Omega; \mathbb{R}^d)$ is the class represented by the cocycle $\mathcal{F}$.

In both the FLC and ILC settings, we denote the class as $[\mathcal{F}]$, or by $[\mathcal{F}(\Omega)]$ when we wish to be explicit about the tiling space in question.

Next we specialize to FLC tiling spaces and construct an invariant in $H^1_s(\Omega; \mathbb{R}^d)$ that classifies homeomorphisms $\Omega \to \Omega'$ up to MLD equivalence and translation.

Definition 5.2. Let $h : \Omega \to \Omega'$ be a homeomorphism of repetitive and FLC tiling spaces. Then $[h] = h^*[\mathcal{F}(\Omega')] \in H^1_s(\Omega, \mathbb{R}^d)$ is called the *cohomology class of h*.

Remark 5.3. Although $[\mathcal{F}(\Omega')]$ is defined via the strongly PE cochain Δx, $h^*(\Delta x)$ is not necessarily PE, and so we cannot do the pullback operation directly in $H^1_{PE,s}$. We must use the isomorphism between $H^1_{PE,s}$ and $\check{H}^1$ to represent $[\mathcal{F}(\Omega')]$ as a class in $\check{H}^1(\Omega', \mathbb{R}^d)$, do the pullback in $\check{H}^1$, and then convert back to $H^1_{PE,s}$.

There is a setting, however, where the pullback can be done directly. Suppose that h sends transversals to transversals. That is, if x is a vertex of T, then $h(T-x)$ has a vertex at the origin. Then PE 0- and 1-cochains on Ω' pull back to PE 0- and 1-cochains on Ω, and we have $[h] = h^*[\Delta x] = [h^*(\Delta x)]$ in $H^1_{PE,s}$.

Example 5.4. Let Ω be a non-periodic and repetitive 1-dimensional tiling space with two types of tiles, A and B, of length L_1 and L_2. (*E.g.*, Ω might be a space of Fibonacci tilings, or Thue-Morse tilings.) Let Ω' be a tiling space with two tiles, A' and B' of length L_1' and L_2', that appear in the exact same sequences as the A and B tiles appear in Ω. Let $h : \Omega \to \Omega'$ be constructed as follows. If T is a tiling in A and B tiles where the origin is a fraction f of the way from the left endpoint of a tile t to the right endpoint, then $h(T)$ is a tiling in A' and B', with the same sequence, in which the origin is a fraction f of the way across the tile t' corresponding to t.

For each tile type t, let i_t be a 1-cochain that evaluates to 1 on each t tile and to 0 on all other tile types, and let $[i_t]$ be the corresponding class

in H^1_{PE}. Under h^*, $i_{A'}$ pulls back to i_A and $i_{B'}$ pulls back to i_B. On Ω', we have $\Delta x = L_1' i_{A'} + L_2' i_{B'}$, so $[h] = [h^* \Delta x] = L_1'[i_A] + L_2'[i_B]$. The difference between $[h]$ and $[\mathcal{F}(\Omega)] = L_1[i_A] + L_2[i_B]$ measures the distortion of lengths that is induced by this homeomorphism.

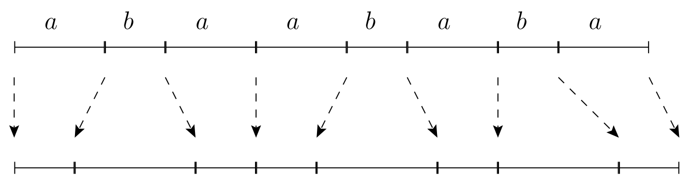

Figure 1: The two tilings have the same combinatorics. Tiles of type a have length 1.5 on top and 1 on the bottom; tiles of type b have length 1 and 2 respectively on top and bottom. The shape-changing deformation induces a piecewise linear map $\mathbb{R} \to \mathbb{R}$, with two different linear parts: either contraction by a factor 2/3, or expansion by a factor 2.

Example 5.5. Let Ω be any FLC tiling space, and let $s : \Omega \to \mathbb{R}^d$ be a weakly PE function. Suppose further that $\|s(T - x_1) - s(T - x_2)\| < \|x_1 - x_2\|$ for all $T \in \Omega$ and $x_{1,2} \in \mathbb{R}^d$. Define $h : \Omega \to \Omega$ by $h(T) = T - s(T)$. Then h is a homeomorphism from Ω to itself, but need not preserve transversals. Thus $h^*(\Delta x)$ is not well-defined. However, $h^*[\mathcal{F}]$ is well-defined, and equals $[\Delta x]$. The reason is that h is homotopic to the identity map, and homotopic maps induce the same pullback map on $\check{H}^1(\Omega, \mathbb{R}^d)$. Since the class of the identity homeomorphism is $[\Delta x]$, so is the class of h.

These two examples may appear special, but in fact *any* homeomorphism can be written as the combination of a translation and a map that sends transversals to transversals.

Proposition 5.6 ([24])**.** *If $h : \Omega \to \Omega'$ is a homeomorphism between FLC tiling spaces, and if Ξ is an FLC transversal of Ω, then there exists $s : \Omega \to \mathbb{R}^d$, continuous and arbitrarily small, such that the map $T \mapsto h(T) - s(T)$ maps Ξ to an FLC transversal of Ω'.*

If we take $h_s(T) = h(T) - s(T)$, then h and h_s are homotopic maps, so $[h] = h^*[\mathcal{F}] = h_s^*[\mathcal{F}] = [h_s^*(\Delta x)]$. In practice, one uses the approximating map h_s to define a pull-back on PE cochains and to actually compute $[h]$, but the class $[h]$ is independent of the approximation.

Example 5.7. Let Ω be a 1-dimensional tiling space coming from the Fibonacci substitution $A \to AB$, $B \to A$, with tile lengths $|A| = \varphi$ and $|B| = 1$, where $\varphi = (1+\sqrt{5})/2$ is the golden mean. Let A_n and B_n denote the patches (called *n-th order supertiles*) obtained by applying the substitution n times to A and B. (*E.g.*, $A_3 = ABAAB$.) Let Ω' be a suspension of the Fibonacci subshift with tile lengths $|A'| = |B'| = (2+\varphi)/(1+\varphi)$, and whose supertiles have lengths $|A'_n|$ and $|B'_n|$. The spaces Ω and Ω' are known to be topologically conjugate [23, 9] via a conjugacy h that preserves the sequence of letters, insofar as the differences in lengths $|A_n| - |A'_n|$ and $|B_n| - |B'_n|$ go to zero as $n \to \infty$.

Since h commutes with translation, one might expect $h^*(\Delta x)$ to equal Δx, and hence for $[h]$ to equal $\mathcal{F}(\Omega)$. However, **this is not the case**. The pullback $h^*(\Delta x)$ is not well-defined, since h does not preserve transversals. To compute $h^*[\mathcal{F}(\Omega')]$ we must find an approximation h_s and compute $[h_s^*(\Delta x)]$.

One such approximation h_{s_0} is given by the procedure of Example 5.4. Tiles A and B get mapped to tiles A' and B', and $h_s^*[\mathcal{F}] = |A'|[i_A] + |B'|[i_b]$. Closer approximations h_{s_n} are obtained by applying the procedure of Example 5.4 to n-th order supertiles, and the map h is the limit of h_{s_n} as $n \to \infty$. Then

$$\begin{aligned}
[h_{s_n}^* \Delta x] &= |A'_n|[i_{A_n}] + |B'_n|[i_{B_n}] \\
&= \begin{pmatrix} |A'| & |B'| \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} [i_{A_n}] \\ [i_{B_n}] \end{pmatrix} \\
&= \begin{pmatrix} |A'| & |B'| \end{pmatrix} \begin{pmatrix} [i_A] \\ [i_B] \end{pmatrix} \\
&= |A'|[i_A] + |B'|[i_B].
\end{aligned}$$

Note that the answer does not depend on n. All maps h_{s_n} are homotopic to each other, and to h itself, so all define the same pullback in cohomology.

6 A cohomological invariant for orbit equivalence

We next study orbit equivalence between minimal, aperiodic tiling spaces that do not necessarily have FLC. An orbit equivalence is a homeomorphism which sends $\mathbb{R}^d$-orbits to $\mathbb{R}^d$-orbits. In dynamics, the term "orbit equivalence" is often used in the framework of $\mathbb{R}$-actions. In that case, it is usually required that the homeomorphism preserves the orientation of the orbits. We make no such assumption here.

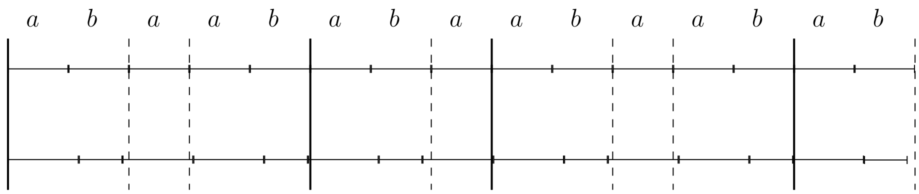

Figure 2: Even though the deformation is not trivial, the global size change of large patches stays bounded: the associated tiling spaces are topologically conjugate.

When a tiling space has FLC, its path components are its translational orbits, so homeomorphisms necessarily preserve orbits. However, this is not true for ILC spaces. For instance, consider a 2-dimensional tiling space Ω (such as the space of pinwheel tilings) that admits continuous rotations. Pick a continuous function f on Ω that is invariant under rotation about the origin. Then consider the map $T \mapsto R_{f(T)}T$, where R_θ is rotation about the origin by an angle θ. If $f(T-x) \neq f(T)$, then it is likely that $T-x$ and T will be sent to different translational orbits.

When two spaces are indeed orbit equivalent, there is a cocycle that describes how an orbit is mapped to another.

Definition 6.1. Given an orbit equivalence $h : \Omega \to \Omega'$, the associated cocycle is the map

$$(T, x) \mapsto h_T(x),$$

defined by $h(T-x) = h(T) - h_T(x)$.

Proposition 6.2. *Given $h : \Omega \to \Omega'$ as above, for all T, the map h_T is a homeomorphism; and $(T, x) \mapsto h_T(x)$ is continuous in two variables. In addition, for all T, the map $\alpha(x, v) = h_T(x+v) - h_T(x)$ is a weakly T-equivariant 1-cocycle in the sense of Section 4.*

Proof. Checking the algebraic cocycle condition on α is straightforward. The non-obvious part is whether $(T, x) \mapsto h_T(x)$ is continuous.

We know that h maps orbits to orbits. The problem is that the topology induced by Ω on orbits (simply by restriction) is not the topology of $\mathbb{R}^d$. However, a standard argument of foliation theory (see for example [20]) guarantees that the orbits (or "leaves") can be given the topology of $\mathbb{R}^d$ without invoking the group action, but using the topology of Ω and its local product structure (see Prop. 3.6).

Given $T \in \Omega$ and $x \in \mathbb{R}^d$, consider γ a path from T to $T-x$. One can choose a "plaque path" from T to $T-x$: a covering of the path γ by

elements of the form $P_i := \varphi_i(\{T_i\} \times B(0, r))$ in local charts. (The sets P_i are called plaques.) Each P_i is given the topology of $\mathbb{R}^n$, which also corresponds to the induced topology by Ω. Then, by uniform continuity of h, and up to reducing the size of the plaques involved, each $h(P_i)$ can be included in an open chart domain V_i of Ω'. Therefore there is a plaque path in Ω': $h(T) \in P'_1, \ldots, P'_k \ni h(T - x) = h(T) - h_T(x)$ with $h(P_i) \subset P'_i$. There is now a plaque path covering $h_*\gamma$. When t tends to 1, $h_*\gamma(t)$ is in P_k, which is homeomorphic to an open set of $\mathbb{R}^d$, so $h_T(\gamma(t))$ tends to $h_T(x)$. Therefore, h_T is continuous at x.

To prove joint continuity in (T, x), fix T_0, x_0 and $\varepsilon > 0$. Let δ be the constant given by uniform continuity of h, corresponding with ε. If T, x are δ_0-close from T_0, x_0 respectively (for an appropriate δ_0), then $T - tx$ and $T_0 - tx_0$ are δ-close from each other for all $t \in [0, 1]$. It means that for all t, $h(T - tx)$ and $h(T_0 - tx_0)$ are ε-close. Now, assume that $h_T(x)$ and $h_{T_0}(x_0)$ are not ε-close. Because $h(T) - h_T(x)$ and a patch of $h(T_0) - h_{T_0}(x_0)$ are ε-close, the underlying Delone sets need to match up to a displacement of ε. By Delone property, it means that if $h_T(x)$ and $h_{T_0}(x_0)$ are not within ε, they must be at least at distance r from each other (r is the uniform separation constant of the Delone set). By continuity of $t \mapsto h_{T_i}(tx)$, it means that there is a t_0 such that $\|h_T(tx) - h_{T_0}(tx_0)\|$ is between, for example, 2ε and $r/2$ (provided ε is small enough). This is a contradiction. $\square$

We will write $[h]$ to denote the cohomology class associated with an orbit equivalence between tiling spaces. *A priori* this is an element of the *weak* group, as this is the only cohomology group that is well-defined in the ILC setting. Even when both Ω and Ω' have FLC, there is no reason why the cocycle should be transversally locally constant with respect to T.

Next suppose that $s : \Omega \to \mathbb{R}^d$ is a continuous function, and that $h_s(T) = h(T) - s(T)$. Then

$$\begin{aligned}
h_s(T - x) &= h(T - x) - s(T - x) \\
&= h(T) - h_T(x) - s(T - x) \\
&= h_s(T) - (h_T(x) + s(T - x) + s(T)) \\
h_{sT}(x) &= h_T(x) + s(T - x) - s(T) \\
\alpha_s(x, v) &= \alpha(x, v) + s(T - (x + v)) - s(T - x).
\end{aligned}$$

That is, in going from h to h_s, the 1-cochain α changes by the coboundary of the weakly PE function s. In particular, $[h_s] = [h]$. This is to be expected, insofar as h_s and h are homotopic.

If the spaces are FLC, and if h_s preserves transversals (as in Proposition 5.6), then α_s defines a class in the *strong* cohomology. Different choices of

s yield homotopic maps, and so define the same class. Thus, whenever the two spaces are FLC, we can define a strong cohomology class by perturbing h to a map h_s that preserves transversals and defining $[h]$ to be $[h_s]$. This is exactly the procedure that we followed (albeit in the PE setting) in the last section.

The following result was proved in [15] in the FLC case. We give here a proof that is valid in both cases of finite and infinite local complexity.

Proposition 6.3. *Let $h : \Omega \to \Omega'$ be an orbit equivalence between two (not necessarily FLC) tiling spaces. Then there exists $\lambda > 1$ and $C > 0$ such that for all $T \in \Omega$ and all $v \in \mathbb{R}^d$,*

$$\lambda^{-1} \|x\| - C \leq \|h_T(x)\| \leq \lambda \|x\| + C.$$

Proof. Consider the cocycle $\alpha : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ defined by $\alpha(T, v) = h_T(v)$. It is continuous, and so the image of the compact set $\Omega \times \overline{B}(0, 1)$ is compact, hence bounded by, say, λ. Let now $v \in \mathbb{R}^d$. We write trivially $v = (n + 1)v/(n + 1)$, where n is the integer part of $\|v\|$. Then for all T (using the cocycle property),

$$\begin{aligned}
\|\alpha(T, v)\| &= \left\| \sum_{i=0}^{n} \alpha(T - i\frac{v}{n+1}, \frac{v}{n+1}) \right\| \\
&\leq \sum_{i=0}^{n} \left\| \alpha(T - i\frac{v}{n+1}, \frac{v}{n+1}) \right\| \\
&\leq (n + 1)\lambda = \lambda \|v\| + \lambda.
\end{aligned}$$

The upper bound of the theorem holds true. For the lower bound, repeat the argument, (replacing h with h^{-1} and noticing that $(h_T)^{-1} = (h^{-1})_{h(T)}$). ☐

We now state and prove the main theorem of this section.

Theorem 6.4. *Let $h_i : \Omega \to \Omega_i$ be two orbit-equivalences ($i \in \{1, 2\}$).*

1. *If $[h_1] = [h_2]$ as elements of the weak cohomology, then there exists a continuous $s : \Omega \to \mathbb{R}^d$ such that $\tau_s : T \mapsto T - s(T)$ is a homeomorphism, and there exists a topological conjugacy $\varphi : \Omega_1 \to \Omega_2$ such that $h_2 \circ \tau_s = \varphi \circ h_1$.*

$$\begin{array}{ccc}
\Omega & \xrightarrow{h_1} & \Omega_1 \\
{\scriptstyle \tau_s}\downarrow & & \downarrow{\scriptstyle \varphi} \\
\Omega & \xrightarrow{h_2} & \Omega_2
\end{array}$$

2. *If the tiling spaces are FLC and $[h_1] = [h_2]$ in the strong cohomology, then the same statement holds, only with φ an MLD map.*

Proof. We begin with the first statement. A simple computation shows that if h is an orbit equivalence between tiling spaces, then $(h^{-1})_{h(T)} = (h_T)^{-1}$. Let h_1 and h_2 be two homeomorphisms whose cochains differ by a co-boundary, *i.e.*, for all T and x,

$$h_{2,T}(x) = h_{1,T}(x) + \big(s_0(T-x) - s_0(T)\big). \tag{2}$$

Now, for $T' \in \Omega_1$ and $w \in \mathbb{R}^d$, let us compute $h_2 \circ h_1^{-1}(T'-w)$. Let $T = h_1^{-1}(T')$.

$$\begin{aligned}
h_2 \circ h_1^{-1}(T'-w) &= h_2\big(T - h_{1,T}^{-1}(w)\big) \\
&= h_2(T) - h_{2,T} \circ h_{1,T}^{-1}(w) \\
&= h_2(T) - h_{1,T} \circ h_{1,T}^{-1}(w) - \Big(s_0\big(T - h_{1,T}^{-1}(w)\big) - s_0(T)\Big) \\
&= h_2 \circ h_1^{-1}(T') - w - s_0\big(h_1^{-1}(T') - (h_1^{-1})_{T'}(w)\big) + s_0\big(h_1^{-1}(T')\big) \\
&= h_2 \circ h_1^{-1}(T') - w - s_0 \circ h_1^{-1}(T'-w) + s_0 \circ h_1^{-1}(T').
\end{aligned}$$

Now, given $T' \in \Omega'$, it is possible to define

$$\varphi(T') = h_2 \circ h_1^{-1}(T') + s_0 \circ h_1^{-1}(T').$$

The computation above shows that $\varphi(T'-w) = \varphi(T') - w$ and in particular it is a bijection on each orbit. Since h_1 and h_2 are bijective, φ induces a bijection between the set of orbits $\Omega/\mathbb{R}^d \to \Omega'/\mathbb{R}^d$. Therefore, φ is bijective. It is continuous and is therefore (by compactness) a homeomorphism. To conclude, φ is a topological conjugacy. A simple computation shows that $h_2^{-1} \circ \varphi \circ h_1 = \tau_s$ with

$$\tau_s(T) = T - (h_{2,T})^{-1}(-s_0(T)),$$

and it is in particular a homeomorphism.

We now turn to the second statement. By functoriality of Čech cohomology, we have

$$(h_2 \circ h_1^{-1})[\mathcal{F}(\Omega_2)]_s = [\mathcal{F}(\Omega_1)]_s,$$

or in other words $[h_2 \circ h_1^{-1}]_s = [\mathcal{F}]_s$. Using the computation from the first part of this theorem, we can choose an arbitrary $T_0 \in \Omega_1$, define

$$\varphi(T_0 - x) = h_2 \circ h_1^{-1}(T_0) - x,$$

and extend it to a topological conjugacy $\Omega_1 \to \Omega_2$. By construction (see above), φ and $h_2 \circ h_1^{-1}$ are homotopic, so $[\varphi]_s = [h_2 \circ h_1^{-1}]_s$. We now want to prove that φ is an MLD map, or in other words that it sends transversals to transversals (Proposition 3.8). Let Ξ be a transversal containing T_0.

Using Proposition 5.6, there exists a local map $\psi : \Omega_1 \to \Omega_2$ that is ε-close to $h_2 \circ h_1^{-1}$, and so homotopic to φ. Without loss of generality, we can also assume that $\psi(T_0) = \varphi(T_0)$. Then there exists a function s on Ω, which is transversally locally constant (in particular its restriction to Ξ is locally constant), such that

$$\psi(T_0 - x) = \psi(T_0) - x + s(T_0 - x) - s(T_0).$$

In particular,

$$\psi(T_0 - x) = \varphi(T_0 - x) + s(T_0 - x) - s(T_0).$$

Let Ξ_0 be a clopen set of Ξ on which s is constant. It means that for any x satisfying $T_0 - x \in \Xi_0$, $\varphi(T_0 - x) = \psi(T_0 - x) - s_0$ for some constant $s_0 \in \mathbb{R}^d$. We deduce:

$$\varphi(\Xi_0) = \psi(\Xi_0) - s_0.$$

Since ψ is local, and the translate of a FLC transversal is a FLC transversal, φ is local. It is therefore a MLD map. $\square$

7 The Ruelle–Sullivan map

In the next two sections, we tackle the inverse problem: "What cohomology classes can one get from a homeomorphism?" We will show that these are precisely the classes that are mapped to invertible matrices by the Ruelle–Sullivan map.

For the moment, we assume that Ω has FLC and is uniquely ergodic with invariant measure μ. In this setting, the notations $[h]$ or $[\alpha]$ will refer to the classes in the *strong* cohomology group associated to a homeomorphism h or a cocycle α.

The Ruelle–Sullivan map for FLC tiling spaces is most easily defined (and was first defined) using the PE de Rham picture for cohomology. See [18] for details. Each class $[\alpha] \in H^1(\Omega; \mathbb{R}^d)$ can be represented by an $\mathbb{R}^d$-valued PE 1-form, *i.e.*, the assignment of a linear transformation $\alpha(T) : \mathbb{R}^d \to \mathbb{R}^d$ at each point $T \in \Omega$. The Ruelle–Sullivan map averages this linear transformation (*i.e.*, $d \times d$ matrix) over Ω with respect to the measure μ.

$$C_\mu([\alpha]) = \int_\Omega \alpha(T) d\mu(T).$$

By the ergodic theorem, this is the same as averaging α over a single orbit, *i.e.*, over the points of a single tiling T. More precisely, we can average over cubes C_r of side r centered at the origin, and take a limit as $r \to \infty$. Adding the differential $d\gamma$ of a bounded function γ can only change the average over C_r by $O(1/r)$, and so does not change the limiting value (see [18]), since

$$\begin{aligned}\int_{C_r} d\gamma(e_1)dx_1 \cdots dx_d &= \int_{C_r} \frac{\partial\gamma}{\partial x_1} dx_1 \cdots dx_d \\ &= \int_{[-\frac{r}{2},\frac{r}{2}]^{d-1}} \gamma(\frac{r}{2}, x_2, \dots x_d) - \gamma(-\frac{r}{2}, x_2, \dots, x_d) d^{d-1}x\end{aligned}$$

scales as r^{d-1}, as does $\int_{C_r} d\gamma(e_j)$ for all $j = 1, 2, \dots, d$. Thus $C_\mu([\alpha])$ is only a function of the cohomology class $[\alpha]$, and not of the specific form α used to represent it.

This is how the Ruelle–Sullivan class was developed for FLC tilings in [18], but the construction never actually uses the FLC condition! A nearly identical construction applies for weak cohomology classes on ILC tiling spaces. The point in both settings is that any (weakly or strongly) PE function or 1-form must be continuous on the compact space Ω, and so must be bounded. The average of α over Ω is always well-defined and by the ergodic theorem can be computed by averaging α over large cubes in a specific tiling T, and adding $d\gamma$ to α can only change the average over C_r by $O(1/r)$.

By the ergodic theorem for uniquely ergodic measures, the convergence of $r^{-d} \int_{C_r} \alpha(T - s)(v)$ to $C_\mu([\alpha])(v)$ is uniform. That is, for any $\varepsilon > 0$ there is a radius r_ε such that, for any tiling T and any centering point x and any vector v, and for any $r > r_\varepsilon$,

$$\left\| r^{-d} \int_{C_r} \alpha(x + s, v) ds - C_\mu([\alpha])(v) \right\| \le \varepsilon \max\{\|v\|, 1\}. \tag{3}$$

Note that this result is uniform in v as well as in x and T. Uniformity for v with $\|v\| \le 2$ follows from compactness of the ball of radius 2. Uniformity for larger vectors follows from the cocycle condition. If $\|v\| > 2$, pick an integer n such that $1 < \|v/n\| < 2$. We can then write

$$\alpha(x + s, v) = \sum_{i=1}^{n} \alpha\Big(x + s + \frac{i-1}{n} v, \frac{v}{n}\Big),$$

and apply equation (3) to each term on the right hand side.

Theorem 7.1. *Let $h : \Omega \to \Omega'$ be a homeomorphism between two tiling spaces (with or without finite local complexity). Then $C_\mu[h]$ is invertible.*

Proof. Let T be fixed, and let $\alpha(x,v) := h_{T-x}(v)$ be the T-equivariant cocycle associated with h. Remember (Proposition 6.3) that α satisfies

$$\lambda^{-1} \|v\| - C \leq \alpha(x,v) \leq \lambda \|x\| + C,$$

for some constants λ and C, uniformly in x. Let $\varepsilon > 0$ be smaller than $\lambda^{-1}/10$. Then, by Equation (3) there is $r = r_\varepsilon$ such that for all x,

$$\left\| r^{-d} \int_{C_r} \alpha(x+s,v)ds - C_\mu(\alpha)(v) \right\| \leq \varepsilon \|v\| .$$

The cocycle property applied to the parallelogram $[x_1, x_1+v, x_2+v, x_2]$ gives:

$$\begin{aligned} \|\alpha(x_1,v) - \alpha(x_2,v)\| &= \|\alpha(x_1+v, x_2-x_1) - \alpha(x_1, x_2-x_1)\| \\ &\leq 2\lambda \|x_2 - x_1\| + 2C. \end{aligned} \tag{4}$$

so that if $x_1, x_2 \in C_r$, the norm $\|\alpha(x_1,v) - \alpha(x_2,v)\|$ is bounded above by $2\lambda\sqrt{d}r + 2C$.

We now combine these results. For all $x_1, x_2 \in C_r$ and all v, we have

$$\|\alpha(x_1,v) - \alpha(x_2,v)\|^2 = \|\alpha(x_1,v)\|^2 + \|\alpha(x_2,v)\|^2 - 2\langle \alpha(x_1,v), \alpha(x_2,v)\rangle,$$

so

$$2\langle \alpha(x_1,v), \alpha(x_2,v)\rangle = \|\alpha(x_1,v)\|^2 + \|\alpha(x_2,v)\|^2 - \|\alpha(x_1,v) - \alpha(x_2,v)\|^2 .$$

The two positive terms on the right hand side are each bounded below by $(\lambda^{-1}\|v\| - C)^2$, and the negative term is bounded below by $-2\sqrt{d}\|x_2 - x_1\| - 2C$. Therefore,

$$\langle \alpha(x_1,v), \alpha(x_2,v)\rangle \geq (\lambda^{-1}\|v\| - C)^2 - c'$$

(where c' depends on r). For all v of norm greater than a constant M (which happens to equal $\lambda(4C + 2\sqrt{C^2 + 3c'})/3$) we can exchange the constant offsets C and c' for a factor of 4:

$$\langle \alpha(x_1,v), \alpha(x_2,v)\rangle \geq \frac{1}{4\lambda^2} \|v\|^2 .$$

In particular, this quantity can be made as large as desired by increasing the length of v. We integrate this quantity over $x_2 \in C_r$, divide by r^d, and find (for all v of norm greater than M):

$$\left\langle \alpha(x_1,v), r^{-d} \int_{C_r} \alpha(s,v)ds \right\rangle \geq \frac{1}{4\lambda^2} \|v\|^2 .$$

Integrating again and taking the square root, we get for all v of norm larger than M:

$$\left\| r^{-d} \int_{C_r} \alpha(s,v) ds \right\| \geq \frac{1}{2\lambda} \|v\| .$$

We conclude this proof by contradiction: assume $C_\mu[h]$ is singular, so that there is $w \in \mathbb{R}^d \setminus \{0\}$ such that $C_\mu[h](w) = 0$. We assume without loss of generality that $\|w\| \geq M$. Then the quantity

$$\left\| r^{-d} \int_{C_r} \alpha(s,w) ds \right\|$$

must be both smaller than $\varepsilon \|w\|$ (with ε very small compared to λ^{-1}), and greater than $1/(2\lambda) \|w\|$, which is a contradiction. Therefore $C_\mu[h]$ is non singular. □

8 Which cohomology classes are achievable?

In the previous section we showed that the strong cohomology class of a homeomorphism of uniquely ergodic FLC tiling spaces (or the weak cohomology class of an orbit-equivalence of uniquely ergodic general tiling spaces) had to be mapped to an invertible matrix by the Ruelle–Sullivan map. The following theorem states that this is the only constraint on the possible cohomology classes of homeomorphisms/orbit equivalences.

Theorem 8.1. *Let Ω be a uniquely ergodic tiling space, and let $[\alpha] \in H^1_w(\Omega;\mathbb{R}^d)$ be such that $C_\mu([\alpha])$ is invertible. Then there exists a tiling space Ω' and a 1-cochain α_0 representing $[\alpha]$, such that a shape deformation induced by α_0 is a homeomorphism*

$$h_{\alpha_0} : \Omega \longrightarrow \Omega'.$$

If Ω also has FLC, and if α represents a class in $H^1_s(\Omega;\mathbb{R}^d)$, then α_0 can be chosen to be strongly pattern-equivariant and Ω' can be chosen to have FLC.

Proof. We begin with the general (ILC) case, and adopt the de Rham picture of cohomology: let T be fixed, and assume ω is a pattern-equivariant closed 1-form which represents $[\alpha]$. Remember that for general tiling spaces, pattern-equivariance means that $T - x \mapsto \omega(x)$ extends to a continuous function on Ω.

Let ρ_r be a continuous function which satisfies the following properties.

- $\rho_r(x)$ is constant for x in the cube $C_r = [-r/2, r/2]^d$, and achieves its maximum value on C_r;
- $\rho_r(x)$ is zero outside of the cube $C_r + [-1, 1]^d$;
- $\rho_r(x) \geq 0$ and $\int_{\mathbb{R}^d} \rho_r = 1$.

The area of the annulus on which ρ_r is not constant grows one order of magnitude slower than the area of C_r, therefore it results from the ergodic theorem that for all $\varepsilon > 0$, there exists r_0 such that for all $r > r_0$ and all x,

$$\left\| \int_{\mathbb{R}^d} \rho_r(s)\omega(x+s)ds - C_\mu[\omega] \right\| \leq \varepsilon \left\| C_\mu[\omega] \right\|, \tag{5}$$

where the norm is any operator norm on the space of $d \times d$ matrices. As before (up to composing with a linear map), we can assume that $C_\mu[\omega]$ is the identity.

Let $\varepsilon < 1/4$ and pick r accordingly so that the equation above holds. Define

$$\overline{\omega}(x) := \int_{\mathbb{R}^d} \rho_r(s)\omega(x+s)ds.$$

It is a continuous, closed 1-form. Since ρ_r has compact support, $\overline{\omega}$ is pattern-equivariant.

Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be the integral of $\overline{\omega}$ (so that $\overline{\omega} = d\varphi$) which satisfies $\varphi(0) = 0$. We want to prove that φ is a homeomorphism of $\mathbb{R}^d$, and that $\overline{\omega}$ and ω define the same cohomology class. Since $\overline{\omega}(x)$ is ε-close to the identity matrix for all x, it is invertible. Therefore, the local inversion theorem states that φ is a local diffeomorphism. The map φ is one-to-one: indeed, assume $\varphi(x) = \varphi(x+v)$, so that

$$\int_x^{x+v} \overline{\omega} = 0.$$

However, $\|\overline{\omega}(s) - I_d\| < 1/4$ for all s. Integrating, we get

$$\left\| \Big(\int_x^{x+v} \overline{\omega} \Big) - v \right\| \leq \int_x^{x+v} \|\overline{\omega} - I_d\| \leq \frac{1}{4} \|v\|,$$

which is a contradiction. In addition, φ is onto: indeed, the image under φ of a large ball $B(0, R)$ contains $B(0, (1-\varepsilon)R)$. As R tends to infinity, this proves surjectivity. Therefore, φ is a global diffeomorphism.

To prove that $\overline{\omega}$ defines the same cohomology class as ω, let us compare:

$$\begin{aligned}\int_0^1 \big[\omega(tv) \cdot v - \overline{\omega}(tv) \cdot v\big] dt &= \int_0^1 \int_{\mathbb{R}^d} \big[\omega(tv) \cdot v - \omega(tv+s) \cdot v\big] \rho_r(s) ds\, dt \\ &= \int_{\mathbb{R}^d} \rho_r(s) \int_0^1 \big[\omega(tv) \cdot v - \omega(tv+s) \cdot v\big] dt\, ds\end{aligned}$$

Since ω is closed, its integral over the boundary of the closed parallelogram $[0, v, v+s, s]$ is zero, therefore

$$\int_0^1 [\omega(tv) \cdot v - \overline{\omega}(tv) \cdot v] dt = \int_{\mathbb{R}^d} \rho_r(s) \int_0^1 [\omega(ts) \cdot s - \omega(v+ts) \cdot s] dt\, ds$$

(compare with equation (4)). The form ω is pattern-equivariant on $\mathbb{R}^d$, so it is bounded (say its matrix norm is bounded by M), and $\|\omega(x) \cdot s\| \leq M \|s\|$ for all x. Since the support of ρ_r is bounded as well, the quantity above is bounded independent of v. Proposition 4.8 guarantees that ω and $\overline{\omega}$ define the same cohomology class.

To conclude, we define a new tiling space Ω' and a map $\Omega \to \Omega'$ as follows. We can assume that T is a (decorated) Delone set which contains 0. Let $T' = \varphi(T)$ (it is a Delone set), and define the decoration on T' by $\ell'(\varphi(x)) := T - x$, so that the set of labels is Ξ, the canonical transversal of Ω.[4] Then, define $h_{\overline{\omega}}(T - x) := T' - \varphi(x)$. It extends to a continuous map from the tiling space of T to the tiling space of T'.

Now suppose that Ω has FLC, and that ω is strongly PE (and so represents a class in $H^1_s(\Omega; \mathbb{R}^d)$). Convolving the s-PE 1-form ω by the function ρ_r of compact support yields another s-PE 1-form $\overline{\omega}$ such that $[\omega] = [\overline{\omega}] \in H^1_s(\Omega; \mathbb{R}^d)$. (If ω is PE with radius R, $\overline{\omega}$ will be PE with radius $R + r + 1$.) We construct Ω' as before, and note that the displacements between two points of a tiling $T' = \varphi(T)$ depends only on the pattern of T in a fixed finite radius around the corresponding points in T. If we label the points of T' by the (finite set of) local patterns of that radius in T, rather than by the entire transversal Ξ, then Ω' has FLC. $\square$

Corollary 8.2. If $h : \Omega \to \Omega'$ is a homeomorphism between FLC, uniquely ergodic tiling spaces, there exists a homeomorphisms h_0 between these two spaces, local in the sense of Definition 3.2, such that $[h]_s = [h_0]_s$. Specifically, there exists a continuous function $s : \Omega \to \mathbb{R}^d$ such that $h(T) = h_0(T) - s(T)$.

Proof. Let $h' : \Omega \to \Omega'$ be a map (in general not a homeomorphism) such that h' maps FLC transversals to FLC transversals (Proposition 5.6). Let $\alpha(T, v) := h'_T(v)$ be the corresponding 1-cocycle. Then $[\alpha] = [h]$ in $H^1_s(\Omega; \mathbb{R}^d)$. If we apply the previous result to α, we obtain a cocycle α_0 representing the same cohomology class, such that for all T, $v \mapsto \alpha_0(T, v)$ is a homeomorphism, and for all v, $T \mapsto \alpha_0(T, v)$ is transversally locally constant (or equivalently, $x \mapsto \alpha_0(T - x, v)$ is strongly T-equivariant). Then,

[4] It is likely possible to pick a much smaller set of labels. We over-decorate here to make sure that the map $\Omega \to \Omega'$ is a homeomorphism and not a factor map.

fix a tiling T_0 and define

$$h_0(T_0 - v) := h'(T_0) - \alpha_0(T_0, v).$$

This map extends to a homeomorphism $\Omega \to \Omega'$. By construction, h_0 agrees with h' on a transversal, say Ξ. Therefore, it sends Ξ to an FLC transversal. Additionally, the function $T \mapsto (h_0)_T(x)$ is transversally locally constant (because α_0 is). Therefore, h_0 sends *any* FLC transversal to an FLC transversal. It is therefore a local map by Proposition 3.8. Finally, $[h_0] = [\alpha_0] = [h]$. □

Corollary 8.3. Within FLC, uniquely ergodic spaces, the equivalence relation "to be homeomorphic" is generated by

- MLD;
- Shape-changing homeomorphisms.

Note that this corollary does *not* say that all homeomorphisms are a combination of MLD maps and shape changes. In fact, a general homeomorphism may also involve translations. However, translations map a space to itself, and do not affect the equivalence relation.

Corollary 8.4. Within the category of uniquely ergodic tiling spaces with or without finite local complexity, the equivalence relation "to be orbit equivalent" is generated by

- Topological conjugacies;
- Continuous shape-changes.

Proof. If $h : \Omega \to \Omega_1$ is an orbit equivalence, then $C_\mu[h]$ is invertible, so by Theorem 8.1 there exists a shape change $h_{\alpha_0} : \Omega \to \Omega_2$ such that $[h_{\alpha_0}] = [h]$ in $H^1_w(\Omega; \mathbb{R}^d)$. But then, by Theorem 6.4, Ω_1 and Ω_2 are topologically conjugate. □

9 Examples

In this section we present some examples to show the difference between the FLC and ILC categories, and the differences between strong and weak cohomology. These differences are already apparent in 1 dimensional tilings.

Recall that the Fibonacci tiling space is generated by the substitution $\sigma(a) = ab, \sigma(b) = a$. A word obtained by applying the substitution σ n

times to a letter is called an n-th order *supertile* and is denoted A_n or B_n. The space of Fibonacci sequences is the space of all bi-infinite words in the letters a and b such that every sub-word is found in A_n or B_n for n sufficiently large. To each sequence we can associate tilings with two types of tiles, ordered in the same way as in the sequence.

Proposition 9.1. *The weak cohomology group H^1_w of the Fibonacci tiling space is infinitely generated over the rationals.*

Proof. We work in pattern-equivariant cohomology. Recall that a closed weakly PE 1-cochain is a weak coboundary if and only if its integral is bounded (Proposition 4.8).

For the Fibonacci tiling space, let ρ_n be a 1-cochain that evaluates to 1 on every tile of each level-n supertile of type a, and to 0 on every tile of each n-supertile of type b. This is strongly PE. Pick $x \in (\varphi - 1, 1)$, where φ is the golden mean. Note that $\sum |x^n|$ converges but $x^n \varphi^n$ goes to infinity. Then $\rho_x := \sum x^n \rho_n$ is a well-defined weakly PE 1-cochain. However ρ_x cannot be a weak coboundary. To see this, consider a level $n+1$ supertile of type a sitting somewhere in a tiling:

$$A_{n+1} = A_n B_n = A_{n-1} B_{n-1} A_{n-1},$$

where the first A_{n-1} is part of an A_n supertile and the second is itself a B_n supertile (see Figure 3). The value of ρ_x on each tile of the first A_{n-1} is exactly x^n greater than the value of ρ_x on the correponding tile of the second A_{n-1}, since they belong to the same m-supertile for all $m > n$, and to corresponding m-supertiles for $m < n$. Since an A_n supertile contains $O(\varphi^n)$ tiles, the integral of ρ_x over the first A_{n-1} is of order $x^n \varphi^n$ greater than the integral over the second. Since $\varphi^n x^n$ is not bounded, and since we can do this comparison for any value of n, it is not possible for ρ_x to have a bounded integral. Hence ρ_x represents a nontrivial class in the weak cohomology of Ω.

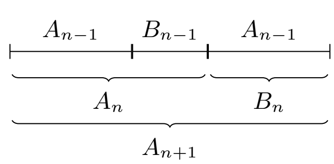

Figure 3: The inclusion of $(n-1)$-supertiles in the n and $(n+1)$-supertiles.

In any finite linear combination $\sum_k c_k \rho_{x_k}$ of such sequences, the term with the largest x will dominate on high-order supertiles, and $\sum_{k,n} c_k x_k^n \varphi^n$

will be unbounded, implying that $\sum_k c_k \rho_{x_k}$ is not a coboundary. Thus the weak cohomology classes for the (uncountably many!) ρ_x's are linearly independent. $\square$

With small modifications, the same construction could be applied to any 1-dimensional hierarchical tiling space.

Now let Ω be the Fibonacci tiling space, say with tile lengths φ and 1, and let Ω' be a deformation of the Fibonacci tiling space by ρ_x for some $x \in (\varphi - 1, 1)$. The following result shows the difference between being homeomorphic to an FLC tiling space and being conjugate to one.

Proposition 9.2. *Ω' is not conjugate to any FLC tiling space.*

Proof. Let $h : \Omega \to \Omega'$ be the deformation map. It is easy to see that $[h]$ is represented by $\beta = \rho_x + \Delta x$. We cannot write β as the sum of a weak coboundary and a strongly PE 1-cochain. To see this, suppose that $\beta = \beta' + d\gamma$, with β' strongly PE with some radius R, and with γ weakly PE (hence bounded). If W_1 and W_2 are two different copies of a high-level supertile within a tiling, then the difference between $\beta(W_1)$ and $\beta(W_2)$ would be bounded, since the integral of $d\gamma$ is bounded and since the values of β' at corresponding points are the same except on regions of length R around each endpoint. However, we have already seen that the integral of ρ_x on different supertiles A_n can differ by arbitrarily large amounts.

Now suppose that $h' : \Omega' \to \Omega''$ is a topological conjugacy and that Ω'' is an FLC tiling space. Then $[h' \circ h] = [h]$ in weak cohomology, and is represented by $\Delta x + \rho_x$. However, $h' \circ h$ is a homeomorphism of FLC tiling spaces, and so $[h' \circ h]$ is a class in the strong cohomology of Ω, and in particular can be represented by a strongly PE 1-cochain. Since $\Delta x + \rho_x$ is not weakly cohomologous to any strongly PE 1-cochain, we have a contradiction. $\square$

The details of the cochain ρ_x are not so important to this argument. The important fact is that the weak cohomology is much bigger than the strong cohomology, so there are many orbit equivalences whose classes are not in the image of the natural map $H^1_s \to H^1_w$. The image of any FLC tiling space by such an orbit equivalence is necessarily an ILC tiling space that is (by construction!) orbit equivalent to an FLC tiling space, but that is not topologically conjugate to any FLC tiling space.

10 Continuous maps between tiling spaces

In this final section, we explain briefly how the results of this paper can be extended beyond orbit equivalences between tiling spaces, to include more general surjective maps that preserve orbits.

For any continuous map $f : \Omega \to \Omega'$ between tiling spaces of finite local complexity, the induced map in Čech cohomology is well defined. We define the class of the map f in $H^1_s(\Omega; \mathbb{R}^d) \simeq \check{H}^1(\Omega; \mathbb{R}^d)$ to be

$$[f]_s := f^*([\mathcal{F}]).$$

Similarly if Ω is a continuous map between arbitrary tiling spaces, such that orbits are mapped into orbits, the equation

$$f(T - v) = f(T) - f_T(v)$$

defines a dynamical 1-cocycle $\alpha(T, v) := f_T(v)$, and the class $[f]_w := [\alpha]$ is well defined in $H^1_w(\Omega; \mathbb{R}^d)$. When f is a homeomorphism of FLC spaces or an orbit equivalence, these definitions agree with our previous notions.

Theorem 10.1. *Let Ω be a tiling space with or without finite local complexity. Assume $h_i : \Omega \to \Omega_i$ are surjective maps to other tiling spaces which preserve orbits, and such that h_1 is an orbit equivalence. If $[h_1]_w = [h_2]_w$, then there exists a factor map $\varphi : \Omega_1 \to \Omega_2$ such that h_2 is homotopic to $\varphi \circ h_1$. If the spaces have FLC and $[h_1]_s = [h_2]_s$, then φ can be chosen to be a local derivation.*

Proof. Much of the proof of Theorem 6.4 carries over. Since $[h_2] = [h_1]$, we must have $h_{2,T}(x) = h_{1,T}(x) + (s_0(T - x) - s_0(T))$ for some continuous function $s_0 : \Omega \to \mathbb{R}^d$. We then compute

$$h_2 \circ h_1^{-1}(T' - w) = h_2 \circ h_1^{-1}(T') - w - s_0 \circ h_1^{-1}(T' - w) + s_0 \circ h_1^{-1}(T'),$$

exactly as before. This computation requires the invertibility of h_1, but makes no assumptions on h_2 beyond the fact that h_2 preserves orbits. Also as before, we define

$$\varphi(T') = h_2 \circ h_1^{-1}(T') + s_0 \circ h_1^{-1}(T'),$$

and see that $\varphi(T' - w) = \varphi(T') - w$. The map φ then extends to a factor map $\Omega_1 \to \Omega_2$. When Ω_1 and Ω_2 have FLC, we check that φ preserves a transversal (exactly as before), making φ a local derivation.

Unraveling the definitions, we see that, for any tiling $T \in \Omega$, $\varphi(h_1(T)) = h_2(T) + s_0(T)$, so

$$h_2(T) = \varphi(h_1(T)) - s_0(T)$$

is homotopic to $\varphi \circ h_1$. □

The only important difference from the proof of Theorem 6.4 is that we cannot write $h_2(T) + s_0(T)$ as a composition $h_2 \circ \tau_s$. Since h_2 is not assumed to be injective on orbits, the map $(h_{2,T})^{-1}$ that we previously used to construct the translation τ_s is no longer well defined.

In the uniquely ergodic case, whenever the class of the map h_2 has a non-singular image under the Ruelle–Sullivan map, the existence of the tiling space Ω_1 and the orbit-equivalence h_1 follow from Theorem 8.1:

Corollary 10.2. If Ω is uniquely ergodic and $h : \Omega \to \Omega'$ is a surjective orbit-preserving map such that $C_\mu[h]$ is non-singular, then h is homotopic to the composition $\varphi \circ h_0$ of a shape deformation h_0 and a factor map φ. In the FLC case, the same statement holds with φ a local derivation.

The result above shows that given an "on average non-singular" map $h : \Omega \to \Omega'$, the lack of bijectivity can have two possible causes. The map h_T can fail to be bijective as a map from $\mathbb{R}^d$ to $\mathbb{R}^d$, in which case h doesn't send a given orbit to its image bijectively. Or h can collapse several orbits into one, in which case φ is a non-invertible factor map.

Factor maps between linearly repetitive tiling spaces have been studied in [10] (see also the review paper [2]). The results of this section suggest that, by separately studying shape deformations and factor maps, we can gain an understanding of how arbitrary tiling spaces are related.

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